Formalized Music

THOUGHT AND MATHEMATICS IN COMPOSITION

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Chapter I

Free Stochastic Music

Art, and above all, music has a fundamental function, which is to catalyze the sublimation that it can bring about through all means of expression. It must aim through fixations which are landmarks to draw towards a total exaltation in which the individual mingles, losing his consciousness in a truth immediate, rare, enormous, and perfect. If a work of art succeeds in this undertaking even for a single moment, it attains its goal. This tremendous truth is not made of objects, emotions, or sensations; it is beyond these, as Beethoven's Seventh Symphony is beyond music. This is why art can lead to realms that religion still occupies for some people.

But this transmutation of every-day artistic material which transforms trivial products into meta-art is a secret. The "possessed" reach it without knowing its "mechanisms." The others struggle in the ideological and technical mainstream of their epoch which constitutes the perishable "climate" and the stylistic fashion. Keeping our eyes fixed on this supreme meta-artistic goal, we shall attempt to define in a more modest manner the paths which can lead to it from our point of departure, which is the magma of contradictions in present music.

There exists a historical parallel between European music and the successive attempts to explain the world by reason. The music of antiquity, causal and deterministic, was already strongly influenced by the schools of Pythagoras and Plato. Plato insisted on the principle of causality, "for it is impossible for anything, to come into being without cause" (*Timaeus*). Strict causality lasted until the nineteenth century when it underwent a

The English translation of Chaps. I–VI is by Christopher A. Butchers.

Fig. I–1. Score of *Metastasis*, 1953/54, Bars 309–17 brutal and fertile transformation as a result of statistical theories in physics. Since antiquity the concepts of chance (*tyche*), disorder (*ataxia*), and disorganization were considered as the opposite and negation of reason (*logos*), order (*taxis*), and organization (*systasis*). It is only recently that knowledge has been able to penetrate chance and has discovered how to separate its degrees—in other words to rationalize it progressively, without, however, succeeding in a definitive and total explanation of the problem of "pure chance."

After a time lag of several decades, atonal music broke up the tonal function and opened up a new path parallel to that of the physical sciences, but at the same time constricted by the virtually absolute determinism of serial music.

It is therefore not surprising that the presence or absence of the principle of causality, first in philosophy and then in the sciences, might influence musical composition. It caused it to follow paths that appeared to be divergent, but which, in fact, coalesced in probability theory and finally in polyvalent logic, which are kinds of generalization and enrichments of the principle of causality. The explanation of the world, and consequently of the sonic phenomena which surround us or which may be created, necessitated and profited from the enlargement of the principle of causality, the basis of which enlargement is formed by the law of large numbers. This law implies an asymptotic evolution towards a stable state, towards a kind of goal, of *stochos*, whence comes the adjective "stochastic."

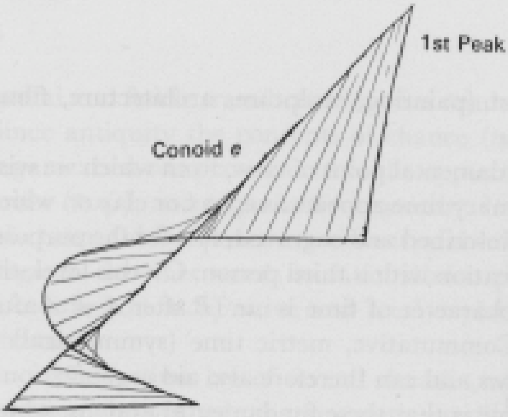
But everything in pure determinism or in less pure indeterminism is subjected to the fundamental operational laws of logic, which were disentangled by mathematical thought under the title of general algebra. These laws operate on isolated states or on sets of elements with the aid of operations, the most primitive of which are the union, notated $\cup$, the intersection, notated $\cap$, and the negation. Equivalence, implication, and quantifications are elementary relations from which all current science can be constructed.

Music, then, may be defined as an organization of these elementary operations and relations between sonic entities or between functions of sonic entities. We understand the first-rate position which is occupied by set theory, not only for the construction of new works, but also for analysis and better comprehension of the works of the past. In the same way a stochastic construction or an investigation of history with the help of stochastics cannot be carried through without the help of logic—the queen of the sciences, and I would even venture to suggest, of the arts—or its mathematical form algebra. For everything that is said here on the subject is also valid for all forms of art (painting, sculpture, architecture, films, etc.).

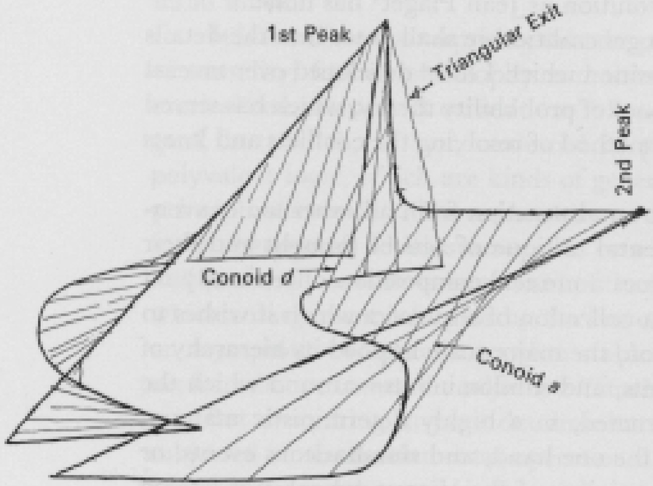
From this very general, fundamental point of view, from which we wish to examine and *make* music, primary time appears as a wax or clay on which operations and relations can be inscribed and engraved, first for the purposes of work, and then for communication with a third person. On this level, the asymmetric, noncommutative character of time is use (B after $A \neq A$ after B, i.e., lexicographic order). Commutative, metric time (symmetrical) is subjected to the same logical laws and can therefore also aid organizational speculations. What is remarkable is that these fundamental notions, which are necessary for construction, are found in man from his tenderest age, and it is fascinating to follow their evolution as Jean Piaget[1] has done.

After this short preamble on generalities we shall enter into the details of an approach to musical composition which I have developed over several years. I call it "stochastic," in honor of probability theory, which has served as a logical framework and as a method of resolving the conflicts and knots encountered.

The first task is to construct an abstraction from all inherited conventions and to exercise a fundamental critique of acts of thought and their materialization. What, in fact, does a musical composition offer strictly on the construction level? It offers a collection of sequences which it wishes to be causal. When, for simplification, the major scale implied its hierarchy of tonal functions—tonics, dominants, and subdominants—around which the other notes gravitated, it constructed, in a highly deterministic manner, linear processes, or melodies on the one hand, and simultaneous events, or chords, on the other. Then the serialists of the Vienna school, not having known how to master logically the indeterminism of atonality, returned to an organization which was extremely causal in the strictest sense, more abstract than that of tonality; however, this abstraction was their great contribution. Messiaen generalized this process and took a great step in systematizing the abstraction of all the variables of instrumental music. What is paradoxical is that he did this in the modal field. He created a multimodal music which immediately found imitators in serial music. At the outset Messiaen's abstract systematization found its most justifiable embodiment in a multiserial music. It is from here that the postwar neo-serialists have drawn their inspiration. They could now, following the Vienna school and Messiaen, with some occasional borrowing from Stravinsky and Debussy, walk on with ears shut and proclaim a truth greater than the others. Other movements were growing stronger; chief among them was the systematic exploration of sonic entities, new instruments, and "noises." Varèse was the

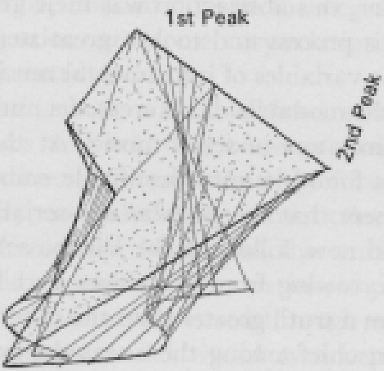


A. Ground profile of the *left half* of the "stomach." The intention was to build a shell, composed of as few ruled surfaces as possible, over the ground plan. A conoid (*e*) is constructed through the ground profile curve ; this wall is bounded by two straight lines : the straight directrix (rising from the left extremity of the ground profile), and the outermost generatrix (passing through the right extremity of the ground profile). This produces the first "peak" of the pavilion.



B. A ruled surface consisting of *two* conoids, *a* and *d*, is laid through the curve bounding the *right half* of the "stomach." The straight directrix of *d* passes through the first peak, and the outermost generatrix at this side forms a triangular exit with the generatrix of *e*. The straight directrix of *a* passes through a second peak and is joined by an arc to the directrix of *d*.

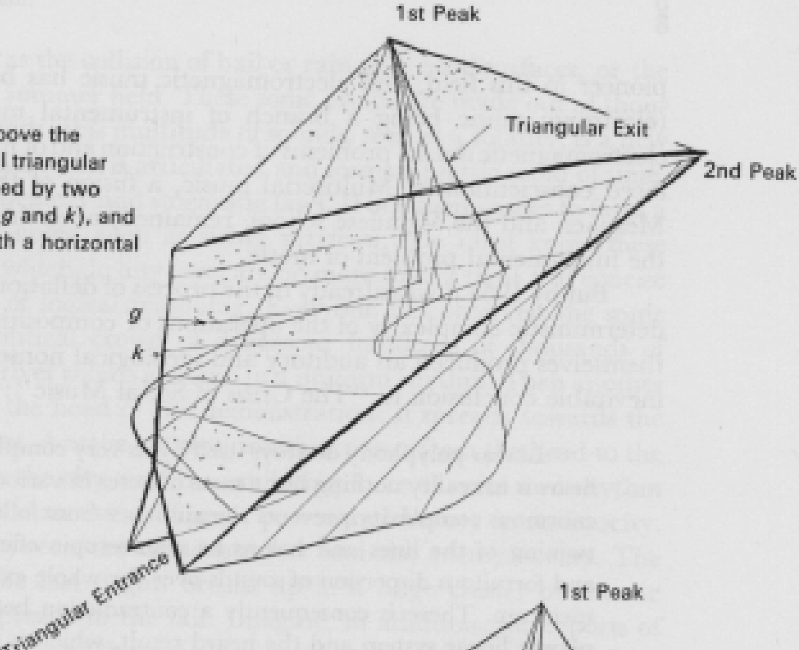
This basic form is the one used in the first design and was retained, with some modifications, in the final structure. The main problem of the design was to establish an aesthetic balance between the two peaks.



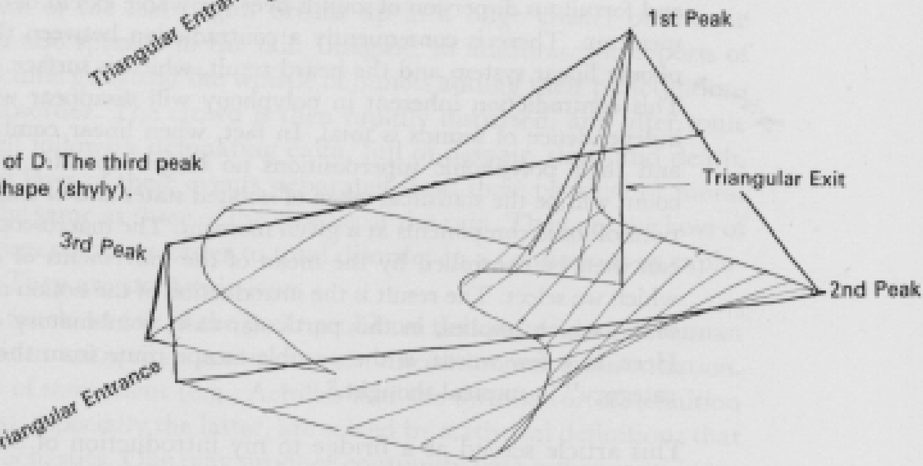
C. Attempt to close the space between the two ruled surfaces of the first design by flat surfaces (which might serve as projection walls).

Fig. I–3. Stages in the Development of the First Design of the Philips Pavilion

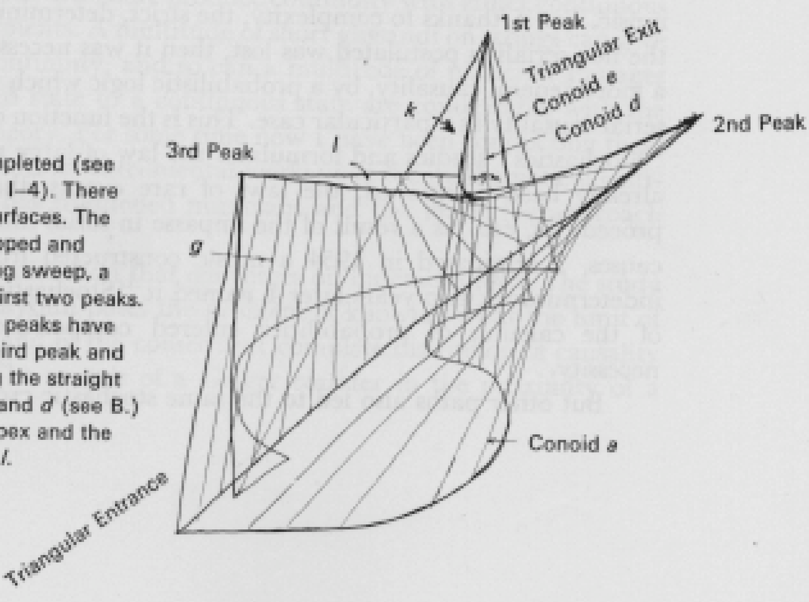
D. Another attempt. Above the entrance channel a small triangular opening is formed, flanked by two hyperbolic paraboloids (*g* and *k*), and the whole is covered with a horizontal top surface.



E. Elaboration of D. The third peak begins to take shape (shyly).



F. The first design completed (see also the first model, Fig. I–4). There are no longer any flat surfaces. The third peak is fully developed and creates, with its opposing sweep, a counterbalance for the first two peaks. The heights of the three peaks have been established. The third peak and the small arc connecting the straight directrixes of conoids *a* and *d* (see B.) form, respectively, the apex and the base of a part of a cone *l*.



pioneer in this field, and electromagnetic music has been the beneficiary (electronic music being a branch of instrumental music). However, in electromagnetic music, problems of construction and of morphology were not faced conscientiously. Multiserial music, a fusion of the multimodality of Messiaen and the Viennese school, remained, nevertheless, at the heart of the fundamental problem of music.

But by 1954 it was already in the process of deflation, for the completely deterministic complexity of the operations of composition and of the works themselves produced an auditory and ideological nonsense. I described the inevitable conclusion in "The Crisis of Serial Music":

> Linear polyphony destroys itself by its very complexity; what one hears is in reality nothing but a mass of notes in various registers. The enormous complexity prevents the audience from following the intertwining of the lines and has as its macroscopic effect an irrational and fortuitous dispersion of sounds over the whole extent of the sonic spectrum. There is consequently a contradiction between the polyphonic linear system and the heard result, which is surface or mass. This contradiction inherent in polyphony will disappear when the independence of sounds is total. In fact, when linear combinations and their polyphonic superpositions no longer operate, what will count will be the statistical mean of isolated states and of transformations of sonic components at a given moment. The macroscopic effect can then be controlled by the mean of the movements of elements which we select. The result is the introduction of the notion of probability, which implies, in this particular case, combinatory calculus. Here, in a few words, is the possible escape route from the "linear category" in musical thought.[2]

This article served as a bridge to my introduction of mathematics in music. For if, thanks to complexity, the strict, deterministic causality which the neo-serialists postulated was lost, then it was necessary to replace it by a more general causality, by a probabilistic logic which would contain strict serial causality as a particular case. This is the function of stochastic science. "Stochastics" studies and formulates the law of large numbers, which has already been mentioned, the laws of rare events, the different aleatory procedures, etc. As a result of the impasse in serial music, as well as other causes, I originated in 1954 a music constructed from the principle of indeterminism; two years later I named it "Stochastic Music." The laws of the calculus of probabilities entered composition through musical necessity.

But other paths also led to the same stochastic crossroads—first of all, natural events such as the collision of hail or rain with hard surfaces, or the song of cicadas in a summer field. These sonic events are made out of thousands of isolated sounds; this multitude of sounds, seen as a totality, is a new sonic event. This mass event is articulated and forms a plastic mold of time, which itself follows aleatory and stochastic laws. If one then wishes to form a large mass of point-notes, such as string pizzicati, one must know these mathematical laws, which, in any case, are no more than a tight and concise expression of chain of logical reasoning. Everyone has observed the sonic phenomena of a political crowd of dozens or hundreds of thousands of people. The human river shouts a slogan in a uniform rhythm. Then another slogan springs from the head of the demonstration; it spreads towards the tail, replacing the first. A wave of transition thus passes from the head to the tail. The clamor fills the city, and the inhibiting force of voice and rhythm reaches a climax. It is an event of great power and beauty in its ferocity. Then the impact between the demonstrators and the enemy occurs. The perfect rhythm of the last slogan breaks up in a huge cluster of chaotic shouts, which also spreads to the tail. Imagine, in addition, the reports of dozens of machine guns and the whistle of bullets adding their punctuations to this total disorder. The crowd is then rapidly dispersed, and after sonic and visual hell follows a detonating calm, full of despair, dust, and death. The statistical laws of these events, separated from their political or moral context, are the same as those of the cicadas or the rain. They are the laws of the passage from complete order to total disorder in a continuous or explosive manner. They are stochastic laws.

Here we touch on one of the great problems that have haunted human intelligence since antiquity: continuous or discontinuous transformation. The sophisms of movement (e.g., Achilles and the tortoise) or of definition (e.g., baldness), especially the latter, are solved by statistical definition; that is to say, by stochastics. One may produce continuity with either continuous or discontinuous elements. A multitude of short glissandi on strings can give the impression of continuity, and so can a multitude of pizzicati. Passages from a discontinuous state to a continuous state are controllable with the aid of probability theory. For some time now I have been conducting these fascinating experiments in instrumental works; but the mathematical character of this music has frightened musicians and has made the approach especially difficult.

Here is another direction that converges on indeterminism. The study of the variation of rhythm poses the problem of knowing what the limit of total asymmetry is, and of the consequent complete disruption of causality among durations. The sounds of a Geiger counter in the proximity of a radioactive source give an impressive idea of this. Stochastics provides the necessary laws.

Before ending this short inspection tour of events rich in the new logic, which were closed to the understanding until recently, I would like to include a short parenthesis. If glissandi are long and sufficiently interlaced, we obtain sonic spaces of continuous evolution. It is possible to produce ruled surfaces by drawing the glissandi as straight lines. I performed this experiment with *Metastasis* (this work had its premiere in 1955 at Donaueschingen). Several years later, when the architect Le Corbusier, whose collaborator I was, asked me to suggest a design for the architecture of the Philips Pavilion in Brussels, my inspiration was pin-pointed by the experiment with *Metastasis*. Thus I believe that on this occasion music and architecture found an intimate connection.[3] Figs. I–1–5 indicate the causal chain of ideas which led me to formulate the architecture of the Philips Pavilion from the score of *Metastasis*.

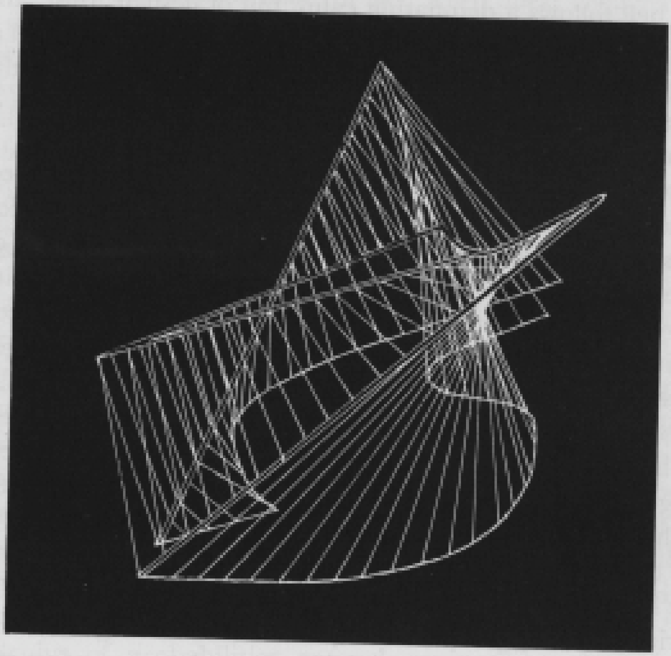


Fig. I–4. First Model of Philips Pavilion

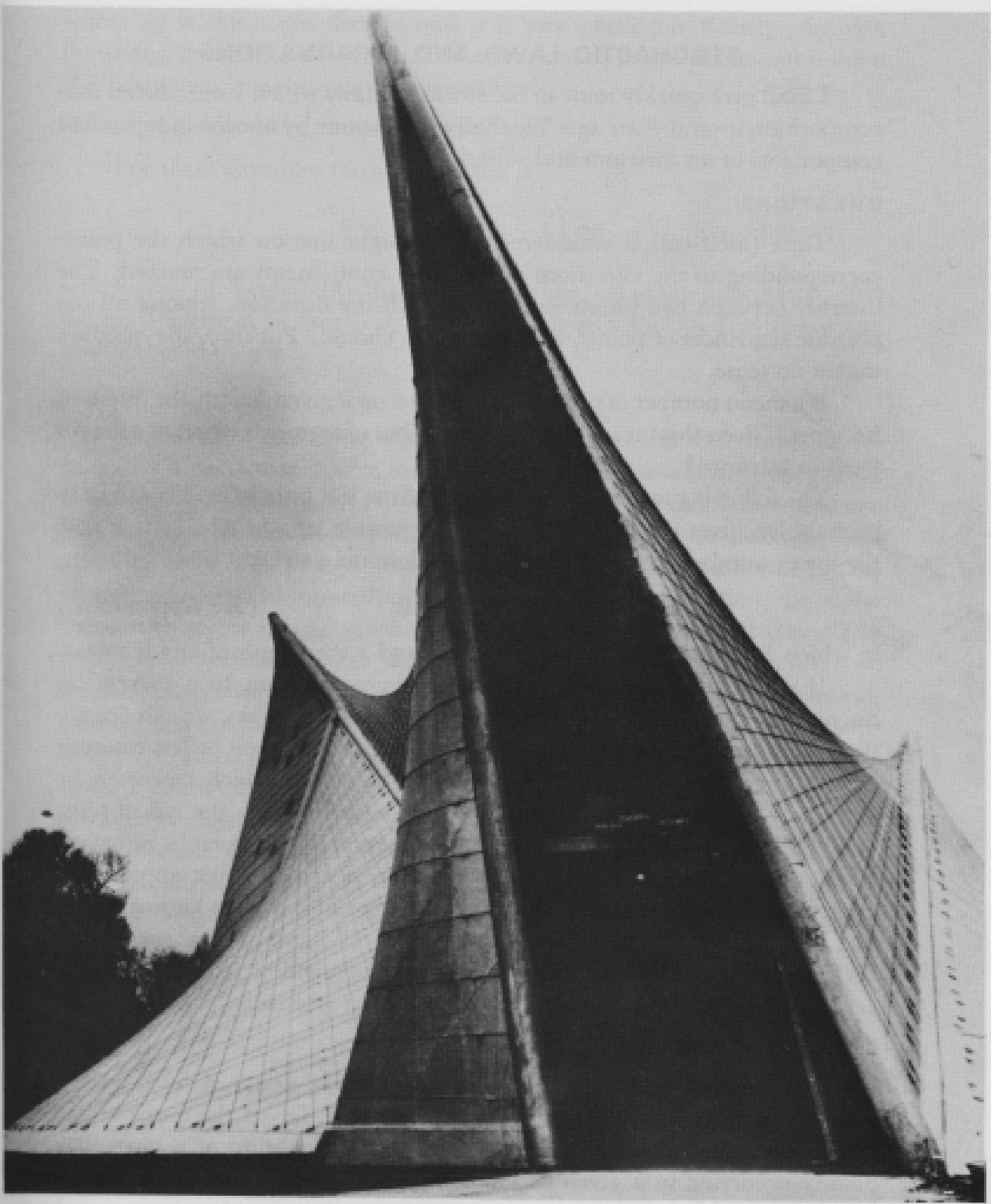


Fig. I–5. Philips Pavilion, Brussels World's Fair, 1958

STOCHASTIC LAWS AND INCARNATIONS

I shall give quickly some of the stochastic laws which I introduced into composition several years ago. We shall examine one by one the independent components of an instrumental sound.

DURATIONS

Time (metrical) is considered as a straight line on which the points corresponding to the variations of the other components are marked. The interval between two points is identical with the duration. Among all the possible sequences of points, which shall we choose? Put thus, the question makes no sense.

If a mean number of points is designated on a given length the question becomes: Given this mean, what is the number of segments equal to a length fixed in advance?

The following formula, which derives from the principles of continuous probability, gives the probabilities for all possible lengths when one knows the mean number of points placed at random on a straight line.

$$P_x = \delta e^{-\delta x}\, dx, \qquad \text{(See Appendix I.)}$$

in which δ is the linear density of points, and x the length of any segment.

If we now choose some points and compare them to a theoretical distribution obeying the above law or any other distribution, we can deduce the amount of chance included in our choice, or the more or less rigorous adaptation of our choice to the law of distribution, which can even be absolutely functional. The comparison can be made with the aid of tests, of which the most widely used is the χ^2 criterion of Pearson. In our case, where all the components of sound can be measured to a first approximation, we shall use in addition the correlation coefficient. It is known that if two populations are in a linear functional relationship, the correlation coefficient is one. If the two populations are independent, the coefficient is zero. All intermediate degrees of relationship are possible.

Clouds of Sounds

Assume a given duration and a set of sound-points defined in the intensity-pitch space realized during this duration. Given the mean superficial density of this tone cluster, what is the probability of a particular density occurring in a given region of the intensity-pitch space? Poisson's Law answers this question:

$$P_\mu = \frac{\mu_0^\mu}{\mu!} e^{-\mu_0},$$

where μ_0 is the mean density and μ is any particular density. As with durations, comparisons with other distributions of sound-points can fashion the law which we wish our cluster to obey.

INTERVALS OF INTENSITY, PITCH, ETC.

For these variables the simplest law is

$$\theta(\gamma)\, d\gamma = \frac{2}{a}\left(1 - \frac{\gamma}{a}\right) d\gamma, \qquad \text{(See Appendix I.)}$$

which gives the probability that a segment (interval of intensity, pitch, etc.) within a segment of length a, will have a length included within γ and $\gamma + d\gamma$, for $0 \leq \gamma \leq a$.

SPEEDS

We have been speaking of sound-points, or granular sounds, which are in reality a particular case of sounds of continuous variation. Among these let us consider glissandi. Of all the possible forms that a glissando sound can take, we shall choose the simplest—the uniformly continuous glissando. This glissando can be assimilated sensorially and physically into the mathematical concept of speed. In a one-dimensional vectorial representation, the scalar size of the vector can be given by the hypotenuse of the right triangle in which the duration and the melodic interval covered form the other two sides. Certain mathematical operations on the continuously variable sounds thus defined are then permitted. The traditional sounds of wind instruments are, for example, particular cases where the speed is zero. A glissando towards higher frequencies can be defined as positive, towards lower frequencies as negative.

We shall demonstrate the simplest logical hypotheses which lead us to a mathematical formula for the distribution of speeds. The arguments which follow are in reality one of those "logical poems" which the human intelligence creates in order to trap the superficial incoherencies of physical phenomena, and which can serve, on the rebound, as a point of departure for building abstract entities, and then incarnations of these entities in sound or light. It is for these reasons that I offer them as examples:

Homogeneity hypotheses [11]*

1. The density of speed-animated sounds is constant; i.e., two regions of equal extent on the pitch range contain the same average number of mobile sounds (glissandi).

* The numbers in brackets correspond to the numbers in the Bibliography at the end of the book.

2. The absolute value of speeds (ascending or descending glissandi) is spread uniformly; i.e., the mean quadratic speed of mobile sounds is the same in different registers.

3. There is isotropy; that is, there is no privileged direction for the movements of mobile sounds in any register. There is an equal number of sounds ascending and descending.

From these three hypotheses of symmetry, we can define the function $f(v)$ of the probability of the absolute speed v. ($f(v)$ is the relative frequency of occurrence of v.)

Let n be the number of glissandi per unit of the pitch range (density of mobile sounds), and r any portion taken from the range. Then the number of speed-animated sounds between v and $v + dv$ and positive, is, from hypotheses 1 and 3:

$$n\, r\, \tfrac{1}{2} f(v)\, dv \quad \text{(the probability that the sign is + is } \tfrac{1}{2}\text{)}.$$

From hypothesis 2 the number of animated sounds with speed of absolute value v is a function which depends on v^2 only. Let this function be $g(v^2)$. We then have the equation

$$n\, r\, \tfrac{1}{2} f(v)\, dv = n\, r\, g(v^2)\, dv.$$

Moreover if $|x| = v$, the probability function $g(v^2)$ will be equal to the law of probability H of x, whence $g(v^2) = H(x)$, or $\log g(v^2) = h(x)$.

In order that $h(x)$ may depend only on $x^2 = v^2$, it is necessary and sufficient that the differentials $d \log g(v^2) = h'(x)\, dx$ and $v\, dv = x\, dx$ have a constant ratio:

$$\frac{d \log g(v^2)}{v\, dv} = \frac{h'(x)\, dx}{x\, dx} = \text{constant} = -2j,$$

whence $h'(x) = -2jx$, $h(x) = -jx^2 + c$, and $H(x) = ke^{-jx^2}$.

But $H(x)$ is a function of elementary probabilities; therefore its integral from $-\infty$ to $+\infty$ must be equal to 1. j is positive and $k = \sqrt{j}/\sqrt{\pi}$. If $j = 1/a^2$, it follows that

$$\tfrac{1}{2} f(v) = g(v^2) = H(x) = \frac{1}{a\sqrt{\pi}}\, e^{-v^2/a^2}$$

and

$$f(v) = \frac{2}{a\sqrt{\pi}}\, e^{-v^2/a^2}$$

for $v = |x|$, which is a Gaussian distribution.

This chain of reasoning borrowed from Paul Lévy was established after Maxwell, who, with Boltzmann, was responsible for the kinetic theory of gases. The function $f(v)$ gives the probability of the speed v; the constant a defines the "temperature" of this sonic atmosphere. The arithmetic mean of v is equal to $a/\sqrt{\pi}$, and the standard deviation is $a/\sqrt{2}$.

We offer as an example several bars from the work *Pithoprakta* for string orchestra (Fig. I–6), written in 1955–56, and performed by Prof. Hermann Scherchen in Munich in March 1957.[4] The graph (Fig. I–7) represents a set of speeds of temperature proportional to $a = 35$. The abscissa represents time in units of 5 cm = 26 MM (Mälzel Metronome). This unit is subdivided into three, four, and five equal parts, which allow very slight differences of duration. The pitches are drawn as the ordinates, with the unit 1 semitone = 0.25 cm. 1 cm on the vertical scale corresponds to a major third. There are 46 stringed instruments, each represented by a jagged line. Each of the lines represents a speed taken from the table of probabilities calculated with the formula

$$f(v) = \frac{2}{a\sqrt{\pi}}e^{-v^2/a^2}.$$

A total of 1148 speeds, distributed in 58 distinct values according to Gauss's law, have been calculated and traced for this passage (measures 52–60, with a duration of 18.5 sec.). The distribution being Gaussian, the macroscopic configuration is a plastic modulation of the sonic material. The same passage was transcribed into traditional notation. To sum up we have a sonic compound in which:

1. The durations do not vary.
2. The mass of pitches is freely modulated.
3. The density of sounds at each moment is constant.
4. The dynamic is *ff* without variation.
5. The timbre is constant.
6. The speeds determine a "temperature" which is subject to local fluctuations. Their distribution is Gaussian.

As we have already had occasion to remark, we can establish more or less strict relationships between the component parts of sounds.[5] The most useful coefficient which measures the degree of correlation between two variables x and y is

$$r = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sqrt{\sum (x - \bar{x})^2}\sqrt{\sum (y - \bar{y})^2}},$$

where $\bar{x}$ and $\bar{y}$ are the arithmetic means of the two variables.

Here then, is the technical aspect of the starting point for a utilization of the theory and calculus of probabilities in musical composition. With the above, we already know that:

1. We can control continuous transformations of large sets of granular and/or continuous sounds. In fact, densities, durations, registers, speeds, etc., can all be subjected to the law of large numbers with the necessary approximations. We can therefore with the aid of means and deviations shape these sets and make them evolve in different directions. The best known is that which goes from order to disorder, or vice versa, and which introduces the concept of entropy. We can conceive of other continuous transformations: for example, a set of plucked sounds transforming continuously into a set of arco sounds, or in electromagnetic music, the passage from one sonic substance to another, assuring thus an organic connection between the two substances. To illustrate this idea, I recall the Greek sophism about baldness: "How many hairs must one remove from a hairy skull in order to make it bald?" It is a problem resolved by the theory of probability with the standard deviation, and known by the term *statistical definition*.

2. A transformation may be explosive when deviations from the mean suddenly become exceptional.

3. We can likewise confront highly improbable events with average events.

4. Very rarified sonic atmospheres may be fashioned and controlled with the aid of formulae such as Poisson's. Thus, even music for a solo instrument can be composed with stochastic methods.

These laws, which we have met before in a multitude of fields, are veritable diamonds of contemporary thought. They govern the laws of the advent of being and becoming. However, it must be well understood that they are not an end in themselves, but marvelous tools of construction and logical lifelines. Here a backfire is to be found. This time it is these stochastic tools that pose a fundamental question: "What is the minimum of logical constraints necessary for the construction of a musical process?" But before answering this we shall sketch briefly the basic phases in the construction of a musical work.

Fig. I–6. Bars 52–57 of *Pithoprakta*

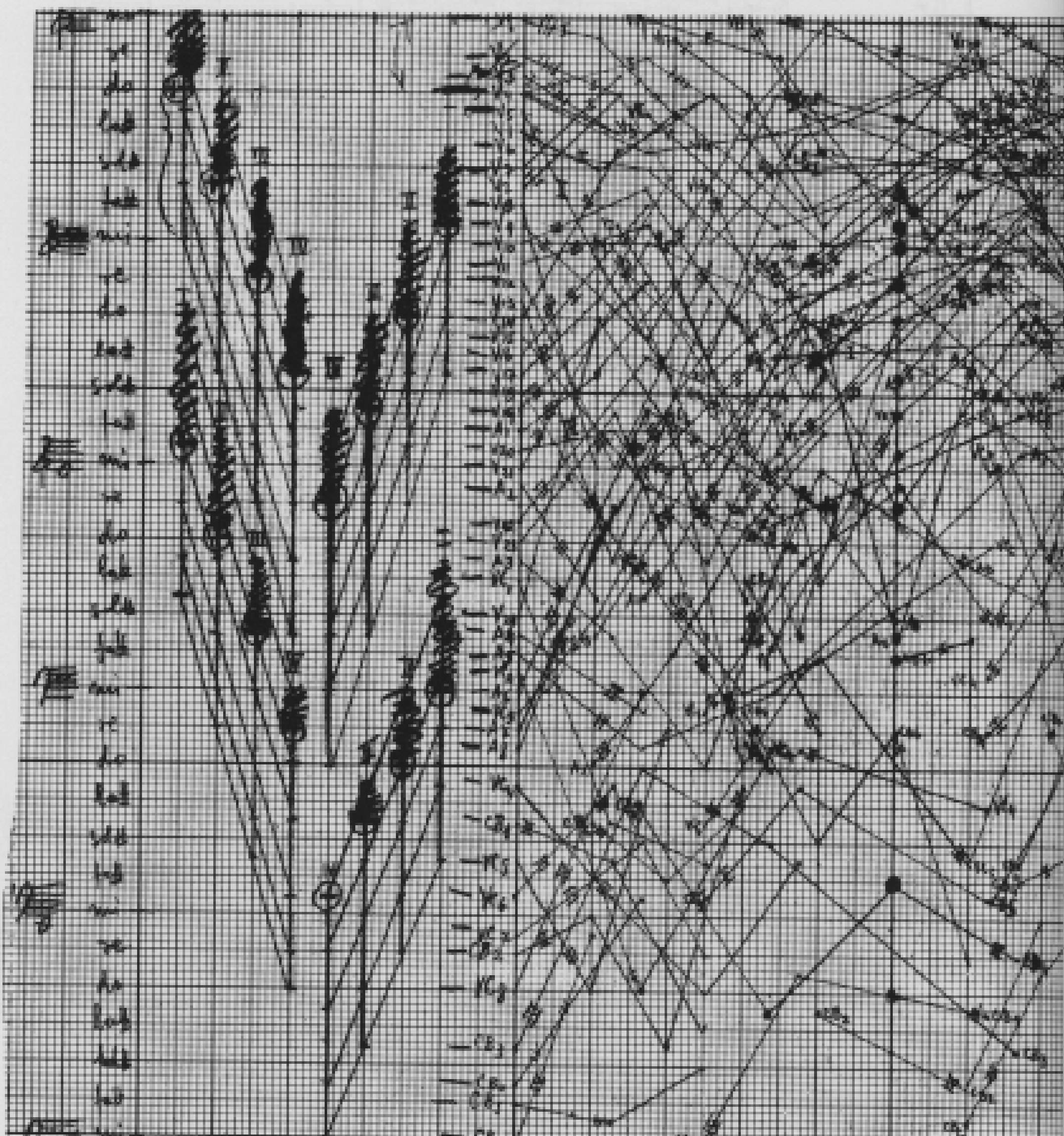
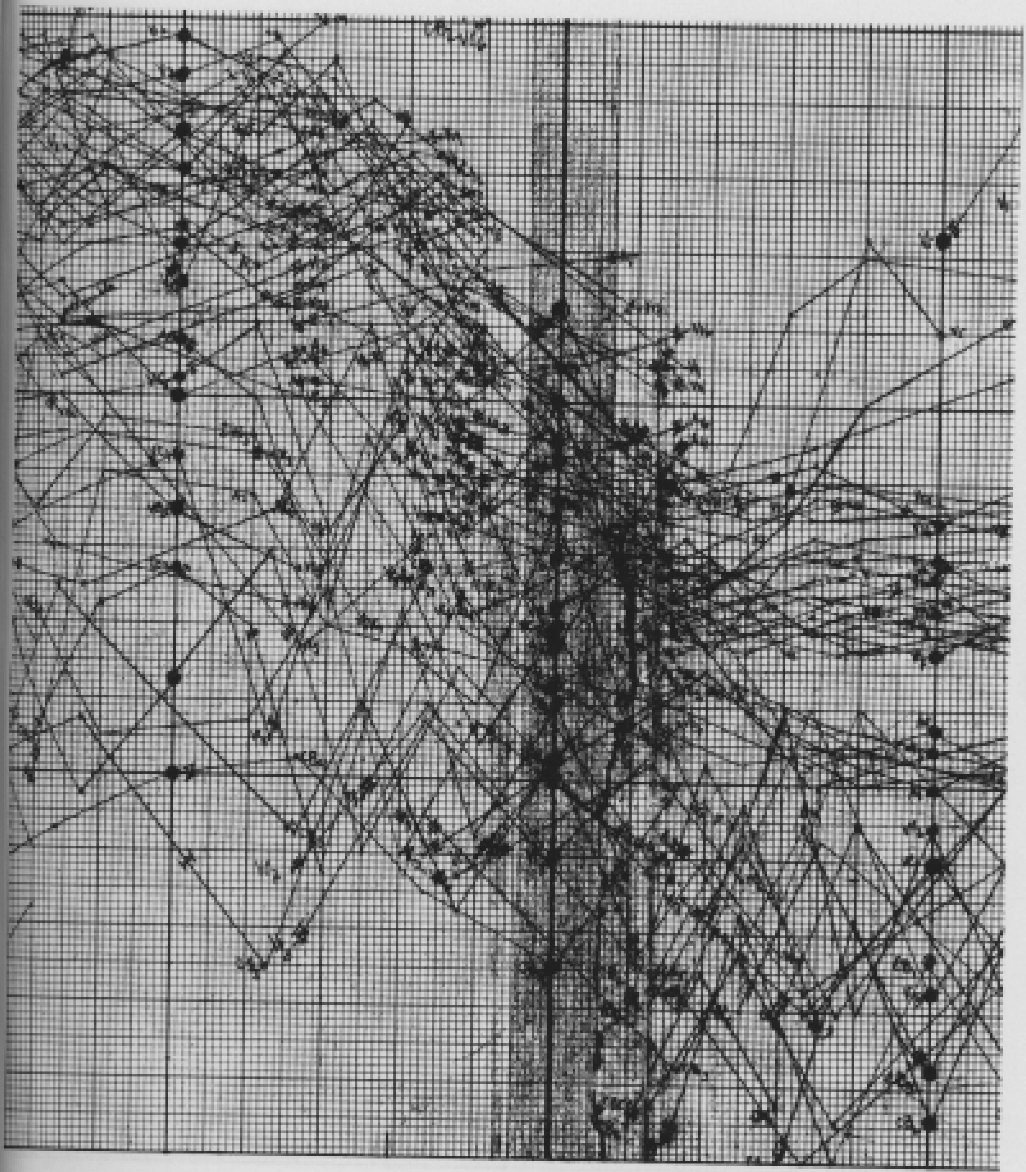


Fig. I–7. Graph of Bars 52–57 of *Pithoprakta*



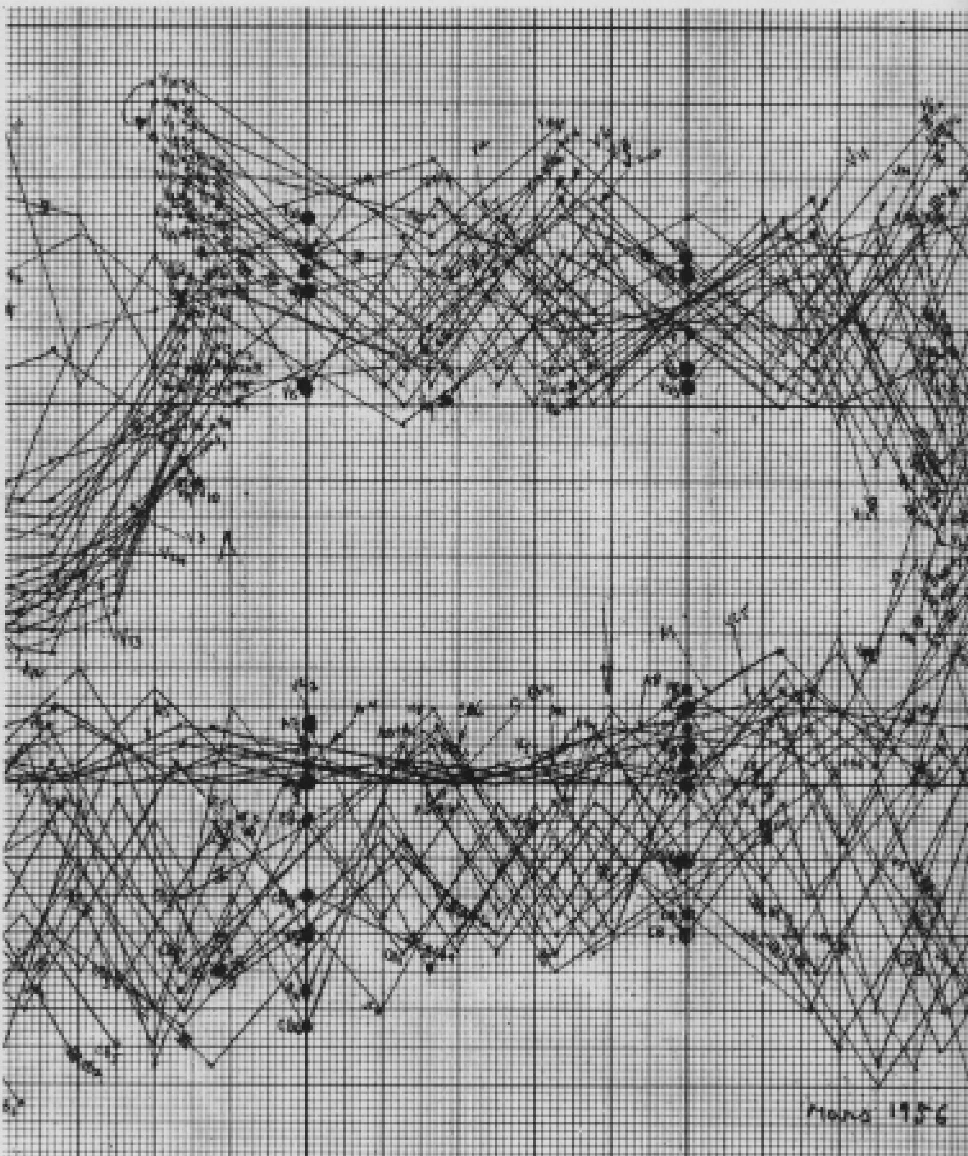


Fig. I–7 *(continued)*